

# Overview of some general results in combinatorial enumeration

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## Abstract

This survey article is devoted to general results in combinatorial enumeration. The first part surveys results on growth of hereditary properties of combinatorial structures. These include permutations, ordered and unordered graphs and hypergraphs, relational structures, and others. The second part advertises five topics in general enumeration: 1. counting lattice points in lattice polytopes, 2. growth of context-free languages, 3. holonomicity (i.e., P-recursiveness) of numbers of labeled regular graphs, 4. frequent occurrence of the asymptotics  $cn^{-3/2}r^n$  and 5. ultimate modular periodicity of numbers of MSOL-definable structures.

## 1 Introduction

We survey some general results in combinatorial enumeration. A *problem* in enumeration is (associated with) an infinite sequence  $P = (S_1, S_2, \dots)$  of finite sets  $S_i$ . Its *counting function*  $f_P$  is given by  $f_P(n) = |S_n|$ , the cardinality of the set  $S_n$ . We are interested in results of the following kind on *general* classes of problems and their counting functions.

**Scheme of general results in combinatorial enumeration.** *The counting function  $f_P$  of every problem  $P$  in the class  $\mathcal{C}$  belongs to the class of functions  $\mathcal{F}$ . Formally,  $\{f_P \mid P \in \mathcal{C}\} \subset \mathcal{F}$ .*

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The larger  $\mathcal{C}$  is, and the more specific the functions in  $\mathcal{F}$  are, the stronger the result. The present overview is a collection of many examples of this scheme.

One can distinguish general results of two types. In *exact results*,  $\mathcal{F}$  is a class of explicitly defined functions, for example polynomials or functions defined by recurrence relations of certain type or functions computable in polynomial time. In *asymptotic results*,  $\mathcal{F}$  consists of functions defined by asymptotic equivalences or asymptotic inequalities, for example functions growing at most exponentially or functions asymptotic to  $n^{(1-1/k)n+o(n)}$  as  $n \rightarrow \infty$ , with the constant  $k \geq 2$  being an integer.

The sets  $S_n$  in  $P$  usually constitute sections of a fixed infinite set. Generally speaking, we take an infinite universe  $U$  of combinatorial structures and introduce problems and classes of problems as subsets of  $U$  and families of subsets of  $U$ , by means of *size functions*  $s : U \rightarrow \mathbb{N}_0 = \{0, 1, 2, \dots\}$  and/or (mostly binary) relations between structures in  $U$ . More specifically, we will mention many results falling within the framework of growth of downsets in partially order sets, or posets.

**Downsets in posets of combinatorial structures.** We consider a nonstrict partial ordering  $(U, \prec)$ , where  $\prec$  is a containment or a substructure relation on a set  $U$  of combinatorial structures, and a size function  $s : U \rightarrow \mathbb{N}_0$ . Problems  $P$  are *downsets* in  $(U, \prec)$ , meaning that  $P \subset U$  and  $A \prec B \in P$  implies  $A \in P$ , and the counting function of  $P$  is

$$f_P(n) = \#\{A \in P \mid s(A) = n\}.$$

(More formally, the problem is the sequence of sections  $(P \cap U_1, P \cap U_2, \dots)$  where  $U_n = \{A \in U \mid s(A) = n\}$ .) Downsets are exactly the sets of the form

$$\text{Av}(F) := \{A \in U \mid A \not\prec B \text{ for every } B \text{ in } F\}, \quad F \subset U.$$

There is a one-to-one correspondence  $P \mapsto F = \min(U \setminus P)$  and  $F \mapsto P = \text{Av}(F)$  between the family of downsets  $P$  and the family of *antichains*  $F$ , which are sets of mutually incomparable structures under  $\prec$ . We call the antichain  $F = \min(U \setminus P)$  corresponding to a downset  $P$  the *base of  $P$* .

We illustrate the scheme by three examples, all for downsets in posets.

## 1.1 Three examples

**Example 1. Downsets of partitions.**  $U$  is the family of partitions of  $[n] = \{1, 2, \dots, n\}$  for  $n$  ranging in  $\mathbb{N}$ , so  $U$  consists of finite sets  $S = \{B_1, B_2, \dots, B_k\}$  of disjoint and nonempty finite subsets  $B_i$  of  $\mathbb{N}$ , called *blocks*, whose union  $B_1 \cup B_2 \cup \dots \cup B_k = [n]$  for some  $n$  in  $\mathbb{N}$ . Two natural size functions on  $U$  are order and size, where the *order*,  $\|S\|$ , of  $S$  is the cardinality,  $n$ , of the underlying set

and the *size*,  $|S|$ , of  $S$  is the number,  $k$ , of blocks. The formula for the number of partitions of  $[n]$  with  $k$  blocks

$$S(n, k) := \#\{S \in U \mid \|S\| = n, |S| = k\} = \sum_{i=0}^k \frac{(-1)^i (k-i)^n}{i!(k-i)!}$$

is a classical result (see [114]);  $S(n, k)$  are called *Stirling numbers*. It is already a simple example of the above scheme but we shall go further.

For fixed  $k$ , the function  $S(n, k)$  is a linear combination with rational coefficients of the exponentials  $1^n, 2^n, \dots, k^n$ . So is the sum  $S(n, 1) + S(n, 2) + \dots + S(n, k)$  counting partitions with order  $n$  and size at most  $k$ . We denote the set of such partitions  $\{S \in U \mid |S| \leq k\}$  as  $U_{\leq k}$ . Consider the poset  $(U, \prec)$  with  $S \prec T$  meaning that there is an *increasing* injection  $f : \bigcup S \rightarrow \bigcup T$  such that every two elements  $x, y$  in  $\bigcup S$  lie in the same block of  $S$  if and only if  $f(x), f(y)$  lie in the same block of  $T$ . In other words,  $S \prec T$  means that  $\bigcup T$  has a subset  $X$  of size  $\|S\|$  such that  $T$  induces on  $X$  a partition order-isomorphic to  $S$ . Note that  $U_{\leq k}$  is a downset in  $(U, \prec)$ . We know that the counting function of  $U_{\leq k}$  with respect to order  $n$  equals  $a_1 1^n + \dots + a_k k^n$  with  $a_i$  in  $\mathbb{Q}$ . What are the counting functions of other downsets? If the size is bounded, as for  $U_{\leq k}$ , they have similar form as shown in the next theorem, proved by Klazar [78]. It is our first example of an exact general enumerative result.

**Theorem 1.1** (Klazar). *If  $P$  is a downset in the poset of partitions such that  $\max_{S \in P} |S| = k$ , then there exist a natural number  $n_0$  and polynomials  $p_1(x)$ ,  $p_2(x)$ ,  $\dots$ ,  $p_k(x)$  with rational coefficients such that for every  $n > n_0$ ,*

$$f_P(n) = \#\{S \in P \mid \|S\| = n\} = p_1(n)1^n + p_2(n)2^n + \dots + p_k(n)k^n.$$

If  $\max_{S \in P} |S| = +\infty$ , the situation is much more intricate and we are far from having a complete description but the growths of  $f_P(n)$  below  $2^{n-1}$  have been determined (see Theorem 2.17 and the following comments). We briefly mention three subexamples of downsets with unbounded size, none of which has  $f_P(n)$  in the form of Theorem 1.1. If  $P$  consists of all partitions of  $[n]$  into intervals of length at most 2, then  $f_P(n) = F_n$ , the  $n^{\text{th}}$  Fibonacci number, and  $f_P(n) = b_1 \alpha^n + b_2 \beta^n$  where  $\alpha = \frac{\sqrt{5}-1}{2}$ ,  $\beta = \frac{\sqrt{5}+1}{2}$  and  $b_1 = \frac{\alpha}{\sqrt{5}}$ ,  $b_2 = \frac{\beta}{\sqrt{5}}$ . If  $P$  is given as  $P = \text{Av}(\{C\})$  where  $C = \{\{1, 3\}, \{2, 4\}\}$  (the partitions in  $P$  are so called *noncrossing partition*, see the survey of Simion [108]) then  $f_P(n) = \frac{1}{n+1} \binom{2n}{n}$ , the  $n^{\text{th}}$  Catalan number which is asymptotically  $cn^{-3/2}4^n$ . Finally, if  $P = U$ , so  $P$  consists of all partitions, then  $f_P(n) = B_n$ , the  $n^{\text{th}}$  Bell number which grows superexponentially.

**Example 2. Hereditary graph properties.**  $U$  is the universe of finite simple graphs  $G = ([n], E)$  with vertex sets  $[n]$ ,  $n$  ranging over  $\mathbb{N}$ , and  $\prec$  is the induced subgraph relation;  $G_1 = ([n_1], E_1) \prec G_2 = ([n_2], E_2)$  means that there is an injection from  $[n_1]$  to  $[n_2]$  (not necessarily increasing) that sends edges to edges and nonedges to nonedges. The size,  $|G|$ , of a graph  $G$  is the number of vertices. Problems are downsets in  $(U, \prec)$  and are called *hereditary graph properties*. The next theorem, proved by Balogh, Bollobás and Weinreich [18], describes counting functions of hereditary graph properties that grow no faster than exponentially.

**Theorem 1.2** (Balogh, Bollobás and Weinreich). *If  $P$  is a hereditary graph property such that for some constant  $c > 1$ ,  $f_P(n) = \#\{G \in P \mid |G| = n\} < c^n$  for every  $n$  in  $\mathbb{N}$ , then there exists a natural number  $n_0$  and polynomials  $p_1(x)$ ,  $p_2(x)$ ,  $\dots$ ,  $p_k(x)$  with rational coefficients such that for every  $n > n_0$ ,*

$$f_P(n) = p_1(n)1^n + p_2(n)2^n + \dots + p_k(n)k^n.$$

The case of superexponential growth of  $f_P(n)$  is discussed below in Theorem 2.11.

In both examples we have the same class of functions  $\mathcal{F}$ , linear combinations  $p_1(n)1^n + p_2(n)2^n + \dots + p_k(n)k^n$  with  $p_i \in \mathbb{Q}[x]$ . It would be nice to find a common extension of Theorems 1.1 and 1.2. It would be also of interest to determine if the two classes of functions realizable as counting functions in both theorems coincide and how they differ from  $\mathbb{Q}[x, 2^x, 3^x, \dots]$ .

**Example 3. Downsets of words.**  $U$  is the set of finite words over a finite alphabet  $A$ , so  $U = \{u = a_1a_2\dots a_k \mid a_i \in A\}$ . The size,  $|u|$ , of such a word is its length  $k$ . The subword relation  $u = a_1a_2\dots a_k \prec v = b_1b_2\dots b_l$  means that  $b_{i+1} = a_1, b_{i+2} = a_2, \dots, b_{i+k} = a_k$  for some  $i$ . We associate with an infinite word  $v = b_1b_2\dots$  over  $A$  the set  $P = P_v$  of all its finite subwords, thus  $P_v = \{b_rb_{r+1}\dots b_s \mid 1 \leq r \leq s\}$ . Note that  $P_v$  is a downset in  $(U, \prec)$ . The next theorem was proved by Morse and Hedlund [94], see also Allouche and Shallit [8, Theorem 10.2.6].

**Theorem 1.3** (Morse and Hedlund). *Let  $P$  be the set of all finite subwords of an infinite word  $v$  over a finite alphabet  $A$ . Then  $f_P(n) = \#\{u \in P \mid |u| = n\}$  is either larger than  $n$  for every  $n$  in  $\mathbb{N}$  or is eventually constant. In the latter case the word  $v$  is eventually periodic.*

The case when  $P$  is a general downset in  $(U, \prec)$ , not necessarily coming from an infinite word (cf. subsection 2.4), is discussed below in Theorem 2.19.

Examples 1 and 2 are exact results and example 3 combines a tight form of an asymptotic inequality with an exact result. Examples 1 and 2 involve only countably many counting functions  $f_P(n)$  and, as follows from the proofs, even only

countably many downsets  $P$ . In example 3 we have uncountably many distinct counting functions. To see this, take  $A = \{0, 1\}$  and consider infinite words  $v$  of the form  $v = 10^{n_1}10^{n_2}10^{n_3}1\dots$  where  $1 \leq n_1 < n_2 < n_3 < \dots$  is a sequence of integers and  $0^m = 00\dots 0$  with  $m$  zeros. It follows that for distinct words  $v$  the counting functions  $f_{P_v}$  are distinct; Proposition 2.1 presents similar arguments in more general settings.

## 1.2 Content of the overview

The previous three examples illuminated to some extent general enumerative results we are interested in but they are not fully representative because we shall cover a larger area than the growth of downsets. We do not attempt to set forth any more formalized definition of a general enumerative result than the initial scheme but in subsections 2.4 and 3.5 we will discuss some general approaches of finite model theory based on relational structures. Not every result or problem mentioned here fits naturally the scheme; Proposition 2.1 and Theorem 2.6 are rather results to the effect that  $\{f_P \mid P \in \mathcal{C}\}$  is too big to be contained in a small class  $\mathcal{F}$ . This collection of general enumerative results is naturally limited by the author's research area and his taste but we do hope that it will be of interest to others and that it will inspire a quest for further generalizations, strengthenings, refinements, common links, unifications etc.

For the lack of space, time and expertise we do not mention results on growth in algebraic structures, especially the continent of growth in groups; we refer the reader for information to de la Harpe [71] (and also to Cameron [45]). Also, this is not a survey on the class of problems  $\#P$  in computational complexity theory (see Papadimitriou [96, Chapter 18]). There are other areas of general enumeration not mentioned properly here, for example 0-1 laws (see Burris [44] and Spencer [111]). Another reason for omissions of nice general results which should be mentioned here is simply the author's ignorance—all suggestions, comments and information will be greatly appreciated.

In the next subsection we review some notions and definitions from combinatorial enumeration, in particular we recall the notion of Wilfian formula (polynomial-time counting algorithm). In Section 2 we review results on growth of downsets in posets of combinatorial structures. Subsection 2.1 is devoted to pattern avoiding permutations, Subsections 2.2 and 2.3 to graphs and related structures, and Subsection 2.4 to relational structures. Most of the results in Subsections 2.2 and 2.3 were found by Balogh and Bollobás and their coauthors ([11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]). We recommend the comprehensive survey of Bollobás [33] on this topic. In Section 3 we advertise five topics in general enumeration together with some related results. 1. The Ehrhart–Macdonald theorem on numbers of lattice

points in lattice polytopes. 2. Growth of context-free languages. 3. The theorem of Gessel on numbers of labeled regular graphs. 4. The theorem of Bell, Burris and Yeats on frequent occurrence of the asymptotics  $cn^{-3/2}r^n$ . 5. The Specker–Blatter theorem on numbers of MSOL-definable structures.

### 1.3 Notation and some specific counting functions

As above, we write  $\mathbb{N}$  for the set  $\{1, 2, 3, \dots\}$ ,  $\mathbb{N}_0$  for  $\{0, 1, 2, \dots\}$  and  $[n]$  for  $\{1, 2, \dots, n\}$ . We use  $\#X$  and  $|X|$  to denote the cardinality of a set. By the phrase “for every  $n$ ” we mean “for every  $n$  in  $\mathbb{N}$ ” and by “for large  $n$ ” we mean “for every  $n$  in  $\mathbb{N}$  with possibly finitely many exceptions”. Asymptotic relations are always based on  $n \rightarrow \infty$ . The *growth constant*  $c = c(P)$  of a problem  $P$  is  $c = \limsup f_P(n)^{1/n}$ ; the reciprocal  $\frac{1}{c}$  is then the radius of convergence of the power series  $\sum_{n \geq 0} f_P(n)x^n$ .

We review several counting sequences appearing in the mentioned results. *Fibonacci numbers*  $(F_n) = (1, 2, 3, 5, 8, 13, \dots)$  are given by the recurrence  $F_0 = F_1 = 1$  and  $F_n = F_{n-1} + F_{n-2}$  for  $n \geq 2$ . They are a particular case  $F_n = F_{n,2}$  of the *generalized Fibonacci numbers*  $F_{n,k}$ , given by the recurrence  $F_{n,k} = 0$  for  $n < 0$ ,  $F_{0,k} = 1$  and  $F_{n,k} = F_{n-1,k} + F_{n-2,k} + \dots + F_{n-k,k}$  for  $n > 0$ . Using the notation  $[x^n]G(x)$  for the coefficient of  $x^n$  in the power series expansion of the expression  $G(x)$ , we have

$$F_{n,k} = [x^n] \frac{1}{1 - x - x^2 - \dots - x^k}.$$

Standard methods provide asymptotic relations  $F_{n,2} \sim c_2(1.618\dots)^n$ ,  $F_{n,3} \sim c_3(1.839\dots)^n$ ,  $F_{n,4} \sim c_4(1.927\dots)^n$  and generally  $F_{n,k} \sim c_k \alpha_k^n$  for constants  $c_k > 0$  and  $1 < \alpha_k < 2$ ;  $\frac{1}{\alpha_k}$  is the least positive root of the denominator  $1 - x - x^2 - \dots - x^k$  and  $\alpha_2, \alpha_3, \dots$  monotonically increase to 2. The unlabeled exponential growth of tournaments (Theorem 2.21) is governed by the *quasi-Fibonacci numbers*  $F_n^*$  defined by the recurrence  $F_0^* = F_1^* = F_2^* = 1$  and  $F_n^* = F_{n-1}^* + F_{n-3}^*$  for  $n \geq 3$ ; so

$$F_n^* = [x^n] \frac{1}{1 - x - x^3}$$

and  $F_n^* \sim c(1.466\dots)^n$ .

We introduced *Stirling numbers*  $S(n, k)$  in example 1. *Bell numbers*  $B_n = \sum_{k=1}^n S(n, k)$  count all partitions of an  $n$ -elements set and follow the recurrence  $B_0 = 1$  and  $B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k$  for  $n \geq 1$ . Equivalently,

$$B_n = [x^n] \sum_{k=0}^{\infty} \frac{x^k}{(1-x)(1-2x)\dots(1-kx)}.$$

The asymptotic form of the Bell numbers is

$$B_n = n^{n(1 - \log \log n / \log n + O(1/\log n))}.$$

The numbers  $p_n$  of *integer partitions* of  $n$  count the ways to express  $n$  as a sum of possibly repeated summands from  $\mathbb{N}$ , with the order of summands being irrelevant. Equivalently,

$$p_n = [x^n] \prod_{k=1}^{\infty} \frac{1}{1 - x^k}.$$

The asymptotic form of  $p_n$  is  $p_n \sim cn^{-1} \exp(d\sqrt{n})$  for some constants  $c, d > 0$ . See Andrews [9] for more information on these asymptotics and for recurrences satisfied by  $p_n$ .

A sequence  $f : \mathbb{N} \rightarrow \mathbb{C}$  is a *quasipolynomial* if for every  $n$  we have  $f(n) = a_k(n)n^k + \cdots + a_1(n)n + a_0(n)$  where  $a_i : \mathbb{N} \rightarrow \mathbb{C}$  are periodic functions. Equivalently,

$$f(n) = [x^n] \frac{p(x)}{(1-x)(1-x^2)\cdots(1-x^l)}$$

for some  $l$  in  $\mathbb{N}$  and a polynomial  $p \in \mathbb{C}[x]$ . We say that the sequence  $f$  is *holonomic* (other terms are P-recursive and D-finite) if it satisfies for every  $n$  (equivalently, for large  $n$ ) a recurrence

$$p_k(n)f(n+k) + p_{k-1}(n)f(n+k-1) + \cdots + p_0(n)f(n) = 0$$

with polynomial coefficients  $p_i \in \mathbb{C}[x]$ , not all zero. Equivalently, the power series  $\sum_{n \geq 0} f(n)x^n$  satisfies a linear differential equation with polynomial coefficients. Holonomic sequences generalize sequences satisfying linear recurrences with constant coefficients. The sequences  $S(n, k)$ ,  $F_{n,k}$ , and  $F_n^*$  for each fixed  $k$  satisfy a linear recurrence with constant coefficients and are holonomic. The sequences of Catalan numbers  $\frac{1}{n+1} \binom{2n}{n}$  and of factorial numbers  $n!$  are holonomic too. The sequences  $B_n$  and  $p_n$  are not holonomic ([115]). It is not hard to show that if  $(a_n)$  is holonomic and every  $a_n$  is in  $\mathbb{Q}$ , then the polynomials  $p_i(x)$  in the recurrence can be taken with integer coefficients. In particular, there are only countably many holonomic rational sequences.

Recall that a power series  $F = \sum_{n \geq 0} a_n x^n$  with  $a_n$  in  $\mathbb{C}$  is *algebraic* if there exists a nonzero polynomial  $Q(x, y)$  in  $\mathbb{C}[x, y]$  such that  $Q(x, F(x)) = 0$ .  $F$  is *rational* if  $Q$  has degree 1 in  $y$ , that is,  $F(x) = R(x)/S(x)$  for two polynomials in  $\mathbb{C}[x]$  where  $S(0) \neq 0$ . It is well known (Comtet [52], Stanley [115]) that algebraic power series have holonomic coefficients and that the coefficients of rational power series satisfy (for large  $n$ ) linear recurrence with constant coefficients.

**Wilfian formulas.** A counting function  $f_P(n)$  has a *Wilfian formula* (Wilf [119]) if there exists an algorithm that calculates  $f_P(n)$  for every input  $n$  effectively, that is to say, in polynomial time. More precisely, we require (extending the definition in [119]) that the algorithm calculates  $f_P(n)$  in the number of steps polynomial in the quantity

$$t = \max(\log n, \log f_P(n))$$

—this is (roughly) the minimum time needed for reading the input and writing down the answer. In the most common situations when  $\exp(n^c) < f_P(n) < \exp(n^d)$  for large  $n$  and some constants  $d > c > 0$ , this amounts to requiring a number of steps polynomial in  $n$ . But if  $f_P(n)$  is small (say  $\log n$ ) or big (say doubly exponential in  $n$ ), then one has to work with  $t$  in place of  $n$ . The class of counting functions with Wilfian formulas includes holonomic sequences but is much more comprehensive than that.

## 2 Growth of downsets of combinatorial structures

We survey results in the already introduced setting of downsets in posets of combinatorial structures  $(U, \prec)$ . The function  $f_P(n)$  counts structures of size  $n$  in the downset  $P$  and  $P$  can also be defined in terms of forbidden substructures as  $P = \text{Av}(F)$ . Besides the containment relation  $\prec$  we employ also isomorphism equivalence relation  $\sim$  on  $U$  and will count *unlabeled* (i.e., nonisomorphic) structures in  $P$ . We denote the corresponding counting function  $g_P(n)$ , so

$$g_P(n) = \#(\{A \in P \mid s(A) = n\} / \sim)$$

is the number of isomorphism classes of structures with size  $n$  in  $P$ .

Restrictions on  $f_P(n)$  and  $g_P(n)$  defining the classes of functions  $\mathcal{F}$  often have the form of *jumps in growth*. A jump is a region of growth prohibited for counting functions—every counting function resides either below it or above it. There are many kinds of jumps but the most spectacular is perhaps the *polynomial–exponential jump* from polynomial to exponential growth, which prohibits counting functions satisfying  $n^k < f_P(n) < c^n$  for large  $n$  for any constants  $k > 0$  and  $c > 1$ . For groups, Grigorchuk constructed a finitely generated group having such intermediate growth (Grigorchuk [69], Grigorchuk and Pak [70], [71]), which excludes the polynomial–exponential jump for general finitely generated groups, but a conjecture says that this jump occurs for every finitely presented group. We



have seen this jump in Theorems 1.1 and 1.2 (from polynomial growth to growth at least  $2^n$ ) and will meet new examples in Theorems 2.4, 2.17, 2.18, 2.21, and 3.3.

If  $(U, \prec)$  has an infinite antichain  $A$ , then under natural conditions we get uncountably many functions  $f_P(n)$ . This was observed several times in the context of permutation containment and for completeness we give the argument here again. These natural conditions, which will always be satisfied in our examples, are *finiteness*, for every  $n$  there are finitely many structures with size  $n$  in  $U$ , and *monotonicity*,  $s(G) \geq s(H)$  &  $G \prec H$  implies  $G = H$  for every  $G, H$  in  $U$ . (Recall that  $G \prec G$  for every  $G$ .)

**Proposition 2.1.** *If  $(U, \prec)$  and the size function  $s(\cdot)$  satisfy the monotonicity and finiteness conditions and  $(U, \prec)$  has an infinite antichain  $A$ , then the set of counting functions  $f_P(n)$  is uncountable.*

**Proof.** By the assumption on  $U$  we can assume that the members of  $A$  have distinct sizes. We show that all the counting functions  $f_{A \vee(F)}$  for  $F \subset A$  are distinct and so this set of functions is uncountable. We write simply  $f_F$  instead of  $f_{A \vee(F)}$ . If  $X, Y$  are two distinct subsets of  $A$ , we express them as  $X = T \cup \{G\} \cup U$  and  $Y = T \cup \{H\} \cup V$  so that, without loss of generality,  $m = s(G) < s(H)$ , and  $G_1 \in T, G_2 \in U$  implies  $s(G_1) < s(G) < s(G_2)$  and similarly for  $Y$  (the sets  $T, U, V$  may be empty). Then, by the assumption on  $\prec$  and  $s(\cdot)$ ,

$$f_X(m) = f_{T \cup \{G\}}(m) = f_T(m) - 1 = f_{T \cup \{H\} \cup V}(m) - 1 = f_Y(m) - 1$$

and  $f_X \neq f_Y$ . □

An infinite antichain thus gives not only uncountably many downsets but in fact uncountably many counting functions. Then, in particular, almost all counting functions are not computable because we have only countably many algorithms. Recently, Albert and Linton [4] significantly refined this argument by showing how certain infinite antichains of permutations produce even uncountably many growth constants, see Theorem 2.6.

On the other hand, if every antichain is finite then there are only countably many functions  $f_P(n)$ . Posets with no infinite antichain are called *well quasiorderings* or shortly *wqo*. (The second part of the wqo property, nonexistence of infinite strictly descending chains, is satisfied automatically by the monotonicity condition.) But even if  $(U, \prec)$  has infinite antichains, there still may be only countably many downsets  $P$  with slow growth of  $f_P(n)$ . For example, this is the case in Theorems 1.1 and 1.2. It is then of interest to determine for which growth uncountably many downsets appear (cf. Theorem 2.5). The posets  $(U, \prec)$  considered here usually have infinite antichains, with two notable wqo exceptions consisting of the minor ordering on graphs and the subsequence ordering on words over a finite alphabet.

## 2.1 Permutations

$U$  is the universe of permutations represented by finite sequences  $b_1b_2\dots b_n$  such that  $\{b_1, b_2, \dots, b_n\} = [n]$ . The size of a permutation  $\pi = a_1a_2\dots a_m$  is its length  $|\pi| = m$ . The containment relation on  $U$  is defined by  $\pi = a_1a_2\dots a_m \prec \rho = b_1b_2\dots b_n$  if and only if for some increasing injection  $f : [m] \rightarrow [n]$  one has  $a_r < a_s \iff b_{f(r)} < b_{f(s)}$  for every  $r, s$  in  $[m]$ . Problems  $P$  are downsets in  $(U, \prec)$  and their counting functions are  $f_P(n) = \#\{\pi \in P \mid |\pi| = n\}$ . The poset of permutations  $(U, \prec)$  has infinite antichains (see Spielman and Bóna [112]). For further information and background on the enumeration of downsets of permutations see Bóna [37].

Recall that  $c(P) = \limsup f_P(n)^{1/n}$ . We define

$$E = \{c(P) \in [0, +\infty] \mid P \text{ is a downset of permutations}\}$$

to be the set of growth constants of downsets of permutations.  $E$  contains elements  $0, 1$  and  $+\infty$  because of the downsets  $\emptyset$ ,  $\{(1, 2, \dots, n) \mid n \in \mathbb{N}\}$  and  $U$  (all permutations), respectively. How much does  $f_P(n)$  drop from  $f_U(n) = n!$  if  $P \neq U$ ? The *Stanley–Wilf conjecture* (Bóna [36, 37]) asserted that it drops to exponential growth. The conjecture was proved in 2004 by Marcus and Tardos [89].

**Theorem 2.2** (Marcus and Tardos). *If  $P$  is a downset of permutations that is not equal to the set of all permutations, then, for some constant  $c$ ,  $f_P(n) < c^n$  for every  $n$ .*

Thus, with the sole exception of  $U$ , every  $P$  has a finite growth constant. Arratia [10] showed that if  $F = \{\pi\}$  then  $c(P) = c(\text{Av}(F))$  is attained as a limit  $\lim f_P(n)^{1/n}$ . It would be nice to extend this result.

**Problem 2.3.** *Does  $\lim f_P(n)^{1/n}$  always exist when  $F$  in  $P = \text{Av}(F)$  has more than one forbidden permutation?*

For infinite  $F$  there conceivably might be oscillations between two different exponential growths (similar oscillations occur for hereditary graph properties and for downsets of words). It would be surprising if oscillations occurred for finite  $F$ .

Kaiser and Klazar [77] determined growths of downsets of permutations in the range up to  $2^{n-1}$ .

**Theorem 2.4** (Kaiser and Klazar). *If  $P$  is a downset of permutations, then exactly one of the four cases occurs.*

1. For large  $n$ ,  $f_P(n)$  is constant.

2. There are integers  $a_0, \dots, a_k$ ,  $k \geq 1$  and  $a_k > 0$ , such that  $f_P(n) = a_0 \binom{n}{0} + \dots + a_k \binom{n}{k}$  for large  $n$ . Moreover,  $f_P(n) \geq n$  for every  $n$ .
3. There are constants  $c, k$  in  $\mathbb{N}$ ,  $k \geq 2$ , such that  $F_{n,k} \leq f_P(n) \leq n^c F_{n,k}$  for every  $n$ , where  $F_{n,k}$  are the generalized Fibonacci numbers.
4. One has  $f_P(n) \geq 2^{n-1}$  for every  $n$ .

The lower bounds in cases 2, 3, and 4 are best possible.

This implies that

$$E \cap [0, 2] = \{0, 1, 2, \alpha_2, \alpha_3, \alpha_4, \dots\},$$

$\alpha_k$  being the growth constants of  $F_{n,k}$ , and that  $\lim f_P(n)^{1/n}$  exists and equals to 0, 1 or to some  $\alpha_k$  whenever  $f_P(n) < 2^{n-1}$  for one  $n$ . Note that 2 is the single accumulation point of  $E \cap [0, 2]$ . We shall see that Theorem 2.4 is subsumed in Theorem 2.17 on ordered graphs. Case 2 and case 3 with  $k = 2$  give the *polynomial–Fibonacci jump*: If  $P$  is a downset of permutations, then either  $f_P(n)$  grows at most polynomially (and in fact equals to a polynomial for large  $n$ ) or at least Fibonaccially. Huczynska and Vatter [73] gave a simpler proof for this jump. Theorem 2.18 extends it to edge-colored cliques. Theorem 2.4 combines an exact result in case 1 and 2 with an asymptotic result in case 3. It would be nice to have in case 3 an exact result too and to determine precise forms of the corresponding functions  $f_P(n)$  (it is known that in cases 1–3 the generating function  $\sum_{n \geq 0} f_P(n)x^n$  is rational, see the remarks at the end of this subsection). Klazar [80] proved that cases 1–3 comprise only countably many downsets, more precisely: if  $f_P(n) < 2^{n-1}$  for one  $n$ , then  $P = \text{Av}(F)$  has finite base  $F$ . In the other direction he showed ([80]) that there are uncountably many downsets  $P$  with  $f_P(n) < (2.336\dots)^n$  for large  $n$ . Recently, Vatter [118] determined the uncountability threshold precisely and extended the description of  $E$  above 2.

**Theorem 2.5** (Vatter). *Let  $\kappa = 2.205\dots$  be the real root of  $x^3 - 2x^2 - 1$ . There are uncountably many downsets of permutations  $P$  with  $c(P) \leq \kappa$  but only countably many of them have  $c(P) < \kappa$  and for each of these  $\lim f_P(n)^{1/n}$  exists. Moreover, the countable intersection*

$$E \cap (2, \kappa)$$

*consists exactly of the largest positive roots of the polynomials in the four families ( $k, l$  range over  $\mathbb{N}$ )*

1.  $3 - x - x^{k+1} - 2x^{k+3} + x^{k+4}$ ,
2.  $1 + 2x - x^2 - x^{k+2} - 2x^{k+4} + x^{k+5}$ ,

$$3. 1 + x^k - x^{k+l} - 2x^{k+l+2} + x^{k+l+3}, \text{ and}$$

$$4. 1 - x^k - 2x^{k+2} + x^{k+3}.$$

The set  $E \cap (2, \kappa)$  has no accumulation point from above but it has infinitely many accumulation points from below:  $\kappa$  is the smallest element of  $E$  which is an accumulation point of accumulation points. The smallest element of  $E \cap (2, \kappa)$  is  $2.065 \dots$  ( $k = l = 1$  in the family 3).

In [13] it was conjectured that all elements of  $E$  (even in the more general situation of ordered graphs) are algebraic numbers and that  $E$  has no accumulation point from above. These conjectures were refuted by Albert and Linton [4]. Recall that a subset of  $\mathbb{R}$  is *perfect* if it is closed and has no isolated point. Due to the completeness of  $\mathbb{R}$  such a set is inevitably uncountable.

**Theorem 2.6** (Albert and Linton). *The set  $E$  of growth constants of downsets of permutations contains a perfect subset and therefore is uncountable. Also,  $E$  contains accumulation points from above.*

The perfect subset constructed by Albert and Linton has smallest element  $2.476 \dots$  and they conjecture that  $E$  contains some real interval  $(\lambda, +\infty)$ . However, a typical downset produced by their construction has infinite base. It seems that the refuted conjectures should have been phrased for finitely based downsets.

**Problem 2.7.** *Let  $E^*$  be the countable subset of  $E$  consisting of the growth constants of finitely based downsets of permutations. Show that every  $\alpha$  in  $E^*$  is an algebraic number and that for every  $\alpha$  in  $E^*$  there is a  $\delta > 0$  such that  $(\alpha, \alpha + \delta) \cap E^* = \emptyset$ .*

We know that  $E^* \cap [0, 2] = E \cap [0, 2]$  and probably (as conjectured in [118]) even  $E^* \cap [0, \kappa) = E \cap [0, \kappa)$ .

We turn to the questions of exact counting. In view of Proposition 2.1 and Theorem 2.6, we restrict to downsets of permutations with finite bases. The next problem goes back to Gessel [67, the final section].

**Problem 2.8.** *Is it true that for every finite set of permutations  $F$  the counting function  $f_{\text{Av}(F)}(n)$  is holonomic?*

All explicit  $f_P(n)$  found so far are holonomic. Zeilberger conjectures ([57]) that  $P = \text{Av}(1324)$  has nonholonomic counting function (see Marinov and Radoičić [90] and Albert et al. [3] for the approaches to counting  $\text{Av}(1324)$ ). We remarked earlier that almost all infinitely based  $P$  have nonholonomic  $f_P(n)$ .

More generally, one may pose (Vatter [117]) the following question.

**Problem 2.9.** *Is it true that for every finite set of permutations  $F$  the counting function  $f_{\text{Av}(F)}(n)$  has a Wilfian formula, that is, can be evaluated by an algorithm in number of steps polynomial in  $n$ ?*

Wilfian formulas were shown to exist for several classes of finitely based downsets of permutations. We refer the reader to Vatter [117] for further information and mention here only one such result due to Albert and Atkinson [1]. Recall that  $\pi = a_1 a_2 \dots a_n$  is a *simple permutation* if  $\{a_i, a_{i+1}, \dots, a_j\}$  is not an interval in  $[n]$  for every  $1 \leq i \leq j \leq n$ ,  $0 < j - i < n - 1$ .

**Theorem 2.10** (Albert and Atkinson). *If  $P$  is a downset of permutations containing only finitely many simple permutations, then  $P$  is finitely based and the generating function  $\sum_{n \geq 0} f_P(n)x^n$  is algebraic and thus  $f_P(n)$  has a Wilfian formula.*

Brignall, Ruškuc and Vatter [43] show that it is decidable whether a downset given by its finite basis contains finitely many simple permutations and Brignall, Huczynska and Vatter [42] extend Theorem 2.10 by showing that many subsets of downsets with finitely many simple permutations have algebraic generating functions as well.

We conclude this subsection by looking back at Theorems 2.4 and 2.5 from the standpoint of effectivity. Let a downset of permutations  $P = \text{Av}(F)$  be given by its finite base  $F$ . Then it is decidable whether  $c(P) < 2$  and (as noted in [118]) for  $c(P) < 2$  the results of Albert, Linton and Ruškuc [5] provide effectively a Wilfian formula for  $f_P(n)$ , in fact, the generating function is rational. Also, it is decidable whether  $f_P(n)$  is a polynomial for large  $n$  ([73], Albert, Atkinson and Brignall [2]). By [118], it is decidable whether  $c(P) < \kappa$  and Vatter conjectures that even for  $c(P) < \kappa$  the generating function of  $P$  is rational.

## 2.2 Unordered graphs

$U$  is the universe of finite simple graphs with normalized vertex sets  $[n]$  and  $\prec$  is the induced subgraph relation. Problems  $P$  are hereditary graph properties, that is, downsets in  $(U, \prec)$ , and  $f_P(n)$  counts the graphs in  $P$  with  $n$  vertices. A more restricted family is *monotone properties*, which are hereditary properties that are closed under taking any subgraph. An even more restricted family consists of *minor-closed classes*, which are monotone properties that are closed under contracting edges. By Proposition 2.1 there are uncountably many counting functions of monotone properties (and hence of hereditary properties as well) because, for example, the set of all cycles is an infinite antichain to subgraph ordering. On the other hand, by the monumental theorem of Robertson and Seymour [102] there

are no infinite antichains in the minor ordering and so there are only countably many minor-closed classes. The following remarkable theorem describes growths of hereditary properties.

**Theorem 2.11** (Balogh, Bollobás, Weinreich, Alekseev, Thomason). *If  $P$  is a proper hereditary graph property then exactly one of the four cases occurs.*

1. *There exist rational polynomials  $p_1(x), p_2(x), \dots, p_k(x)$  such that  $f_P(n) = p_1(n) + p_2(n)2^n + \dots + p_k(n)k^n$  for large  $n$ .*
2. *There is a constant  $k$  in  $\mathbb{N}$ ,  $k \geq 2$ , such that  $f_P(n) = n^{(1-1/k)n+o(n)}$  for every  $n$ .*
3. *One has  $n^{n+o(n)} < f_P(n) < 2^{o(n^2)}$  for every  $n$ .*
4. *There is a constant  $k$  in  $\mathbb{N}$ ,  $k \geq 2$ , such that  $f_P(n) = 2^{(1/2-1/2k)n^2+o(n^2)}$  for every  $n$ .*

We mentioned case 1 as Theorem 1.2. The first three cases were proved by Balogh, Bollobás and Weinreich in [18]. The fourth case is due to Alekseev [6] and independently Bollobás and Thomason [34].

Now we will discuss further strengthenings and refinements of Theorem 2.11. Scheinerman and Zito in a pioneering work [105] obtained its weaker version. They showed that for a hereditary graph property  $P$  either (i)  $f_P(n)$  is constantly 0, 1 or 2 for large  $n$  or (ii)  $an^k < f_P(n) < bn^k$  for every  $n$  and some constants  $k$  in  $\mathbb{N}$  and  $0 < a < b$  or (iii)  $n^{-c}k^n \leq f_P(n) \leq n^ck^n$  for every  $n$  and constants  $c, k$  in  $\mathbb{N}$ ,  $k \geq 2$ , or (iv)  $n^{cn} \leq f_P(n) \leq n^{dn}$  for every  $n$  and some constants  $0 < c < d$  or (v)  $f_P(n) > n^{cn}$  for large  $n$  for every constant  $c > 0$ .

In cases 1, 2, and 4 growths of  $f_P(n)$  settle to specific asymptotic values and these can be characterized by certain minimal hereditary properties, as shown in [18]. Case 3, the penultimate rate of growth ([19]), is very different. Balogh, Bollobás and Weinreich proved in [19] that for every  $c > 1$  and  $\varepsilon > 1/c$  there is a monotone property  $P$  such that

$$f_P(n) \in [n^{cn+o(n)}, 2^{(1+o(1))n^{2-\varepsilon}}]$$

for every  $n$  and  $f_P(n)$  attains either extremity of the interval infinitely often. Thus in case 3 the growth may oscillate (infinitely often) between the bottom and top parts of the range. The paper [19] contains further examples of oscillations (we stated here just one simplified version) and a conjecture that for finite  $F$  the functions  $f_{\text{Av}(F)}(n)$  do not oscillate. As for the upper boundary of the range, in [19] it is proven that for every monotone property  $P$ ,

$$f_P(n) = 2^{o(n^2)} \Rightarrow f_P(n) < 2^{n^{2-1/t+o(1)}} \text{ for some } t \text{ in } \mathbb{N}.$$

For hereditary properties this jump is only conjectured. What about the lower boundary? The paper [21] is devoted to the proof of the following theorem.

**Theorem 2.12** (Balogh, Bollobás and Weinreich). *If  $P$  is a hereditary graph property, then exactly one of the two cases occurs.*

1. *There is a constant  $k$  in  $\mathbb{N}$  such that  $f_P(n) < n^{(1-1/k)n+o(n)}$  for every  $n$ .*
2. *For large  $n$ , one has  $f_P(n) \geq B_n$  where  $B_n$  are Bell numbers. This lower bound is the best possible.*

By the theorem, the growth of Bell numbers is the lower boundary of the penultimate growth in case 3 of Theorem 2.11.

Monotone properties of graphs are hereditary and therefore their counting functions follow Theorem 2.11. Their more restricted nature allows for simpler proofs and simple characterizations of minimal monotone properties, which is done in the paper [20]. Certain growths of hereditary properties do not occur for monotone properties, for example if  $P$  is monotone and  $f_P(n)$  is unbounded, then  $f_P(n) \geq \binom{n}{2} + 1$  for every  $n$  ([20]) but,  $P$  consisting of complete graphs with possibly an additional isolated vertex is a hereditary property with  $f_P(n) = n + 1$  for  $n \geq 3$ . More generally, Balogh, Bollobás and Weinreich show in [20] that if  $P$  is monotone and  $f_P(n)$  grows polynomially, then

$$f_P(n) = a_0 \binom{n}{0} + a_1 \binom{n}{1} + \cdots + a_k \binom{n}{k} \quad \text{for large } n$$

and some integer constants  $0 \leq a_j \leq 2^{j(j-1)/2}$ . In fact, [20] deals mostly with general results on the extremal functions  $e_P(n) := \max\{|E| \mid G = ([n], E) \in P\}$  for monotone properties  $P$ .

For the top growths in case 4 of Theorem 2.11, Alekseev [6] and Bollobás and Thomason [35] proved that for  $P = \text{Av}(F)$  with  $f_P(n) = 2^{(1/2-1/2k)n^2+o(n^2)}$  the parameter  $k$  is equal to the maximum  $r$  such that there is an  $s$ ,  $0 \leq s \leq r$ , with the property that no graph in  $F$  can have its vertex set partitioned into  $r$  (possibly empty) blocks inducing  $s$  complete graphs and  $r - s$  empty graphs. For monotone properties this reduces to  $k = \min\{\chi(G) - 1 \mid G \in F\}$  where  $\chi$  is the usual chromatic number of graphs. Balogh, Bollobás and Simonovits [17] replaced for monotone properties the error term  $o(n^2)$  by  $O(n^\gamma)$ ,  $\gamma = \gamma(F) < 2$ . Ishigami [75] recently extended case 4 to  $k$ -uniform hypergraphs.

Minor-closed classes of graphs were recently looked at from the point of view of counting functions as well. They again follow Theorem 2.11, with possible simplifications due to their more restricted nature. One is that there are only countably many minor-closed classes. Another simplification is that, with the

trivial exception of the class of all graphs, case 4 does not occur as proved by Norine et al. [95].

**Theorem 2.13** (Norine, Seymour, Thomas and Wollan). *If  $P$  is a proper minor-closed class of graphs then  $f_P(n) < c^n n!$  for every  $n$  in  $\mathbb{N}$  for a constant  $c > 1$ .*

Bernardi, Noy and Welsh [28] obtained the following theorem; we shorten its statement by omitting characterizations of classes  $P$  with the given growth rates.

**Theorem 2.14** (Bernardi, Noy and Welsh). *If  $P$  is a proper minor-closed class of graphs then exactly one of the six cases occurs.*

1. *The counting function  $f_P(n)$  is constantly 0 or 1 for large  $n$ .*
2. *For large  $n$ ,  $f_P(n) = p(n)$  for a rational polynomial  $p$  of degree at least 2.*
3. *For every  $n$ ,  $2^{n-1} \leq f_P(n) < c^n$  for a constant  $c > 2$ .*
4. *There exist constants  $k$  in  $\mathbb{N}$ ,  $k \geq 2$ , and  $0 < a < b$  such that  $a^n n^{(1-1/k)n} < f_P(n) < b^n n^{(1-1/k)n}$  for every  $n$ .*
5. *For every  $n$ ,  $B_n \leq f_P(n) = o(1)^n n!$  where  $B_n$  is the  $n^{\text{th}}$  Bell number.*
6. *For every  $n$ ,  $n! \leq f_P(n) < c^n n!$  for a constant  $c > 1$ .*

*The lower bounds in cases 3, 5 and 6 are best possible.*

In fact, in case 3 the formulas of Theorem 1.2 apply. Using the strongly restricted nature of minor-closed classes, one could perhaps obtain in case 3 an even more specific exact result. Paper [28] gives further results on the growth constants  $\lim (f_P(n)/n!)^{1/n}$  in case 6 and states several open problems, of which we mention the following analogue of Theorem 2.2 for unlabeled graphs. Similar conjecture was stated also in McDiarmid, Steger and Welsh [91].

**Problem 2.15.** *Does every proper minor-closed class of graphs contain at most  $c^n$  nonisomorphic graphs on  $n$  vertices, for a constant  $c > 1$ ?*

This brings us to the unlabeled count of hereditary properties. The following theorem was obtained by Balogh et al. [16].

**Theorem 2.16** (Balogh, Bollobás, Saks and Sós). *If  $P$  is a hereditary graph property and  $g_P(n)$  counts nonisomorphic graphs in  $P$  by the number of vertices, then exactly one of the three cases occurs.*

1. *For large  $n$ ,  $g_P(n)$  is constantly 0, 1 or 2.*



2. For every  $n$ ,  $g_P(n) = cn^k + O(n^{k-1})$  for some constants  $k$  in  $\mathbb{N}$  and  $c$  in  $\mathbb{Q}$ ,  $c > 0$ .
3. For large  $n$ ,  $g_P(n) \geq p_n$  where  $p_n$  is the number of integer partitions of  $n$ . This lower bound is best possible.

(We have shortened the statement by omitting the characterizations of  $P$  with given growth rates.) The authors of [16] remark that with more effort case 2 can be strengthened, for large  $n$ , to an exact result with the error term  $O(n^{k-1})$  replaced by a quasipolynomial  $p(n)$  of degree at most  $k-1$ . It turns out that a weaker form of the jump from case 2 to case 3 was proved already by Macpherson [87, 88]: If  $G = (\mathbb{N}, E)$  is an infinite graph and  $g_G(n)$  is the number of its unlabeled  $n$ -vertex induced subgraphs then either  $g_G(n) \leq n^c$  for every  $n$  and a constant  $c > 0$  or  $g_G(n) > \exp(n^{1/2-\varepsilon})$  for large  $n$  for every constant  $\varepsilon > 0$ . Pouzet [98] showed that in the former case  $c_1 n^d < g_G(n) < c_2 n^d$  for every  $n$  and some constants  $0 < c_1 < c_2$  and  $d$  in  $\mathbb{N}$ .

## 2.3 Ordered graphs and hypergraphs, edge-colored cliques, words, posets, tournaments, and tuples

**Ordered graphs.** As previously,  $U$  is the universe of finite simple graphs with vertex sets  $[n]$  but  $\prec$  is now the *ordered* induced subgraph relation, which means that  $G_1 = ([m], E_1) \prec G_2 = ([n], E_2)$  if and only if there is an increasing injection  $f : [m] \rightarrow [n]$  such that  $\{u, v\} \in E_1 \iff \{f(u), f(v)\} \in E_2$ . Problems are downsets  $P$  in  $(U, \prec)$ , are called *hereditary properties of ordered graphs*, and  $f_P(n)$  is the number of graphs in  $P$  with vertex set  $[n]$ . The next theorem, proved by Balogh, Bollobás and Morris [13], vastly generalizes Theorem 2.4.

**Theorem 2.17** (Balogh, Bollobás and Morris). *If  $P$  is a hereditary property of ordered graphs, then exactly one of the four cases occurs.*

1. For large  $n$ ,  $f_P(n)$  is constant.
2. There are integers  $a_0, \dots, a_k$ ,  $k \geq 1$  and  $a_k > 0$ , such that  $f_P(n) = a_0 \binom{n}{0} + \dots + a_k \binom{n}{k}$  for large  $n$ . Moreover,  $f_P(n) \geq n$  for every  $n$ .
3. There are constants  $c, k$  in  $\mathbb{N}$ ,  $k \geq 2$ , such that  $F_{n,k} \leq f_P(n) \leq n^c F_{n,k}$  for every  $n$ , where  $F_{n,k}$  are the generalized Fibonacci numbers.
4. One has  $f_P(n) \geq 2^{n-1}$  for every  $n$ .

The lower bounds in cases 2, 3, and 4 are best possible.

This is an extension of Theorem 2.4 because the poset of permutations is embedded in the poset of ordered graphs via representing a permutation  $\pi = a_1 a_2 \dots a_n$  by the graph  $G_\pi = ([n], \{\{i, j\} \mid i < j \text{ \& } a_i < a_j\})$ . One can check that  $\pi \prec \rho \iff G_\pi \prec G_\rho$  and that the graphs  $G_\pi$  form a downset in the poset of ordered graphs, so Theorem 2.17 implies Theorem 2.4. Similarly, the poset of set partitions of example 1 in Introduction is embedded in the poset of ordered graphs, via representing partitions by graphs whose components are cliques. Thus the growths of downsets of set partitions in the range up to  $2^{n-1}$  are described by Theorem 2.17. As for permutations, it would be nice to have in case 3 an exact result. Balogh, Bollobás and Morris conjecture that  $2.031\dots$  (the largest real root of  $x^5 - x^4 - x^3 - x^2 - 2x - 1$ ) is the smallest growth constant for ordered graphs above 2 and Vatter ([118]) notes that this is not an element of  $E$  and thus here the growth constants for permutations and for ordered graphs part ways.

**Edge-colored cliques.** Klazar [82] considered the universe  $U$  of pairs  $(n, \chi)$  where  $n$  ranges over  $\mathbb{N}$  and  $\chi$  is a mapping from the set  $\binom{[n]}{2}$  of two-element subsets of  $[n]$  to a finite set of colors  $C$ . The containment  $\prec$  is defined by  $(m, \phi) \prec (n, \chi)$  if and only if there is an increasing injection  $f : [m] \rightarrow [n]$  such that  $\chi(\{f(x), f(y)\}) = \phi(\{x, y\})$  for every  $x, y$  in  $[m]$ . For two colors we recover ordered graphs with induced ordered subgraph relation. In [82] the following theorem was proved.

**Theorem 2.18** (Klazar). *If  $P$  is a downset of edge-colored cliques, then exactly one of the three cases occurs.*

1. *The function  $f_P(n)$  is constant for large  $n$ .*
2. *There is a constant  $c$  in  $\mathbb{N}$  such that  $n \leq f_P(n) \leq n^c$  for every  $n$ .*
3. *One has  $f_P(n) \geq F_n$  for every  $n$ , where  $F_n$  are the Fibonacci numbers.*

*The lower bounds in cases 2 and 3 are best possible.*

This extends the bounded-linear jump and the polynomial-Fibonacci jump of Theorem 2.17. It would be interesting to have full Theorem 2.17 in this more general setting. As explained in [82], many posets of structures can be embedded in the poset of edge-colored cliques (as we have just seen for permutations) and thus Theorem 2.18 applies to them. With more effort, case 2 can be strengthened to the exact result  $f_P(n) = p(n)$  with rational polynomial  $p(x)$ .

**Words over finite alphabet.** We revisit example 3 from Introduction. Recall that  $U = A^*$  consists of all finite words over finite alphabet  $A$  and  $\prec$  is the subword ordering. This ordering has infinite antichains, for example  $11, 101, 1001, \dots$  for  $A = \{0, 1\}$ . Balogh and Bollobás [11] investigated general downsets in  $(A^*, \prec)$  and proved the following extension of Theorem 1.3.

**Theorem 2.19** (Balogh and Bollobás). *If  $P$  is a downset of finite words over a finite alphabet  $A$  in the subword ordering, then  $f_P(n)$  is either bounded or  $f_P(n) \geq n + 1$  for every  $n$ .*

In contrast with Theorem 1.3, for general downsets, a bounded function  $f_P(n)$  need not be eventually constant. Balogh and Bollobás [11] showed that for fixed  $s$  in  $\mathbb{N}$  function  $f_P(n)$  may oscillate infinitely often between the maximum and minimum values  $s^2$  and  $2s - 1$ , and  $s^2 + s$  and  $2s$ . These are, however, the wildest bounded oscillations possible since they proved, as their main result, that if  $f_P(n) = m \leq n$  for some  $n$  then  $f_P(N) \leq (m + 1)^2/4$  for every  $N$ ,  $N \geq n + m$ . They also gave examples of unbounded oscillations of  $f_P(n)$  between  $n + g(n)$  and  $2^{n/g(n)}$  for any increasing and unbounded function  $g(n) = o(\log n)$ , with the downset  $P$  coming from an infinite word over two-letter alphabet.

Another natural ordering on  $A^*$  is the *subsequence ordering*,  $a_1 a_2 \dots a_k \prec b_1 b_2 \dots b_l$  if and only if  $b_{i_1} = a_1, b_{i_2} = a_2, \dots, b_{i_k} = a_k$  for some indices  $1 \leq i_1 < i_2 < \dots < i_k \leq l$ . Downsets in this ordering remain downsets in the subword ordering and thus their counting functions are governed by Theorem 2.19. But they can be also embedded in the poset of edge-colored complete graphs (associate with  $a_1 a_2 \dots a_n$  the pair  $(n, \chi)$  where  $\chi(\{i, j\}) = \{a_i, a_j\}$  for  $i < j$ ) and Theorem 2.18 applies. In particular, if  $P \subset A^*$  is a downset in the subsequence ordering, then  $f_P(n)$  is constant for large  $n$  or  $f_P(n) \geq n + 1$  for every  $n$  (by Theorems 2.18 and 2.19). The subsequence ordering on  $A^*$  is a wqo by Higman's theorem ([72]) and therefore has only countably many downsets.

A variation on the subsequence ordering is the ordering on  $A^*$  given by  $u = a_1 a_2 \dots a_k \prec v = b_1 b_2 \dots b_l$  if and only if there is a permutation  $\pi$  of the alphabet  $A$  such that  $a_1 a_2 \dots a_k \prec \pi(b_1) \pi(b_2) \dots \pi(b_l)$  in the subsequence ordering, that is,  $u$  becomes a subsequence of  $v$  after the letters in  $v$  are injectively renamed. This ordering on  $A^*$  gives example 1 in Introduction and leads to Theorem 1.1. It is a wqo as well.

**Posets and tournaments.**  $U$  is the set of all pairs  $S = ([n], \leq_S)$  where  $\leq_S$  is a non-strict partial ordering on  $[n]$ . We set  $R = ([m], \leq_R) \prec S = ([n], \leq_S)$  if and only if there is an injection  $f : [m] \rightarrow [n]$  such that  $x \leq_R y \iff f(x) \leq_S f(y)$  for every  $x, y$  in  $[m]$ . Thus  $R \prec S$  means that the poset  $R$  is an induced subposet of  $S$ . Downsets in  $(U, \prec)$ , *hereditary properties of posets*, and their growths were investigated by Balogh, Bollobás and Morris in [15]. For the unlabeled count they obtained the following result.

**Theorem 2.20** (Balogh, Bollobás and Morris). *If  $P$  is a hereditary property of posets and  $g_P(n)$  counts nonisomorphic posets in  $P$  by the number of vertices, then exactly one of the three cases occurs.*

1. Function  $g_P(n)$  is bounded.
2. There is a constant  $c > 0$  such that, for every  $n$ ,  $\lceil \frac{n+1}{2} \rceil \leq g_P(n) \leq \lceil \frac{n+1}{2} \rceil + c$ .
3. For every  $n$ ,  $g_P(n) \geq n$ .

The lower bounds in cases 2 and 3 are best possible.

As for the labeled count  $f_P(n)$ , using case 1 of Theorem 2.11 they proved ([15, Theorem 2]) that if  $P$  is a hereditary property of posets then either (i)  $f_P(n)$  is constantly 1 for large  $n$  or (ii) there are  $k$  integers  $a_1, \dots, a_k$ ,  $a_k \neq 0$ , such that  $f_P(n) = a_0 \binom{n}{0} + \dots + a_k \binom{n}{k}$  for large  $n$  or (iii)  $f_P(n) \geq 2^n - 1$  for every  $n$ ,  $n \geq 6$ . Moreover, the lower bound in case (iii) is best possible and in case (ii) one has  $f_P(n) \geq \binom{n}{0} + \dots + \binom{n}{k}$  for every  $n$ ,  $n \geq 2k + 1$  and this bound is also best possible.

A *tournament* is a pair  $T = ([n], T)$  where  $T$  is a binary relation on  $[n]$  such that  $xTx$  for no  $x$  in  $[n]$  and for every two distinct elements  $x, y$  in  $[n]$  exactly one of  $xTy$  and  $yTx$  holds.  $\mathcal{U}$  consists of all tournaments for  $n$  ranging in  $\mathbb{N}$  and  $\prec$  is the induced subtournament relation. Balogh, Bollobás and Morris considered in [14, 15] unlabeled counting functions of hereditary properties of tournaments. We merge their results in a single theorem.

**Theorem 2.21** (Balogh, Bollobás and Morris). *If  $P$  is a hereditary property of tournaments and  $g_P(n)$  counts nonisomorphic tournaments in  $P$  by the number of vertices, then exactly one of the three cases occurs.*

1. For large  $n$ , function  $g_P(n)$  is constant.
2. There are constants  $k$  in  $\mathbb{N}$  and  $0 < c < d$  such that  $cn^k < g_P(n) < dn^k$  for every  $n$ . Moreover,  $g_P(n) \geq n - 2$  for every  $n$ ,  $n \geq 4$ .
3. For every  $n$ ,  $n \neq 4$ , one has  $g_P(n) \geq F_n^*$  where  $F_n^*$  are the quasi-Fibonacci numbers.

The lower bounds in cases 2 and 3 are best possible.

Case 1 and the second part of case 2 were proved in [15] and the rest of the theorem in [14]. A closely related and in one direction stronger theorem was independently obtained by Boudabbous and Pouzet [38] (see also [99, Theorem 22]): If  $g_T(n)$  counts unlabeled  $n$ -vertex subtournaments of an infinite tournament  $T$ , then either  $g_T(n)$  is a quasipolynomial for large  $n$  or  $g_T(n) > c^n$  for large  $n$  for a constant  $c > 1$ .

**Ordered hypergraphs.**  $\mathcal{U}$  consists of all hypergraphs, which are the pairs  $H = ([n], H)$  with  $n$  in  $\mathbb{N}$  and  $H$  being a set of nonempty and non-singleton

subsets of  $[n]$ , called edges. Note that  $U$  extends both the universe of finite simple graphs and the universe of set partitions. The containment  $\prec$  is ordered but non-induced and is defined by  $([m], G) \prec ([n], H)$  if and only if there is an increasing injection  $f : [m] \rightarrow [n]$  and an injection  $g : G \rightarrow H$  such that for every edge  $E$  in  $G$  we have  $f(E) \subset g(E)$ . Equivalently, one can omit some vertices from  $[n]$ , some edges from  $H$  and delete some vertices from the remaining edges in  $H$  so that the resulting hypergraph is order-isomorphic to  $G$ . Downsets in  $(U, \prec)$  are called *strongly monotone properties of ordered hypergraphs*. Again,  $f_P(n)$  counts hypergraphs in  $P$  with the vertex set  $[n]$ . We define a special downset  $\Pi$ : we associate with every permutation  $\pi = a_1 a_2 \dots a_n$  the (hyper)graph  $G_\pi = ([2n], \{\{i, n + a_i\} \mid i \in [n]\})$  and let  $\Pi$  denote the set of all hypergraphs in  $U$  contained in some graph  $G_\pi$ ; the graphs in  $\Pi$  differ from  $G_\pi$ 's only in adding in all ways isolated vertices. Note that  $\pi \prec \rho$  for two permutations if and only if  $G_\pi \prec G_\rho$  for the corresponding (hyper)graphs. The next theorem was conjectured by Klazar in [78] for set partitions and in [79] for ordered hypergraphs.

**Theorem 2.22** (Balogh, Bollobás, Morris, Klazar, Marcus). *If  $P$  is a strongly monotone property of ordered hypergraphs, then exactly one of the two cases occurs.*

1. *There is a constant  $c > 1$  such that  $f_P(n) \leq c^n$  for every  $n$ .*
2. *One has  $P \supset \Pi$ , which implies that*

$$f_P(n) \geq \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} k! = n^{n+O(n/\log n)}$$

*for every  $n$  and that the lower bound is best possible.*

The theorem was proved by Balogh, Bollobás and Morris [12] and independently by Klazar and Marcus [83] (by means of results from [81, 79, 89]). It follows that Theorem 2.22 implies Theorem 2.2. Theorem 2.22 was motivated by efforts to extend the Stanley–Wilf conjecture, now the Marcus–Tardos theorem, from permutations to more general structures. A further extension would be to have it for the wider class of hereditary properties of ordered hypergraphs. These correspond to the containment  $\prec$  defined by  $([m], G) \prec ([n], H)$  if and only if there is an increasing injection  $f : [m] \rightarrow [n]$  such that  $\{f(E) \mid E \in G\} = \{f([m]) \cap E \mid E \in H\}$ . The following conjecture was proposed in [12].

**Problem 2.23.** *If  $P$  is a hereditary property of ordered hypergraphs, then either  $f_P(n) \leq c^n$  for every  $n$  for some constant  $c > 1$  or one has  $f_P(n) \geq \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} k!$  for every  $n$ . Moreover, the lower bound is best possible.*

As noted in [12], now it is no longer true that in the latter case  $P$  must contain  $\Pi$ .

**Tuples of nonnegative integers.** For a fixed  $k$  in  $\mathbb{N}$ , we set  $U = \mathbb{N}_0^k$ , so  $U$  contains all  $k$ -tuples of nonnegative integers. We define the containment by  $a = (a_1, a_2, \dots, a_k) \prec b = (b_1, b_2, \dots, b_k)$  if and only if  $a_i \leq b_i$  for every  $i$ . By Higman's theorem,  $(U, \prec)$  is wqo. So there are only countably many downsets. The size function  $\|\cdot\|$  on  $U$  is given by  $\|a\| = a_1 + a_2 + \dots + a_k$ . For a downset  $P$  in  $(U, \prec)$ ,  $f_P(n)$  counts all tuples in  $P$  whose entries sum up to  $n$ . Stanley [113] (see also [114, Exercise 6 in Chapter 4]) obtained the following result.

**Theorem 2.24** (Stanley). *If  $P$  is a downset of  $k$ -tuples of nonnegative integers, then there is a rational polynomial  $p(x)$  such that  $f_P(n) = p(n)$  for large  $n$ .*

In fact, the theorem holds for upsets as well because they are complements of downsets and  $\#\{a \in \mathbb{N}_0^k \mid \|a\| = n\} = \binom{n+k-1}{k-1}$  is a rational polynomial in  $n$ . Jelínek and Klazar [76] noted that the theorem holds for the larger class of sets  $P \subset \mathbb{N}_0^k$  that are finite unions of the generalized orthants  $\{a \in \mathbb{N}_0^k \mid a_i \geq_i b_i, i \in [k]\}$ , here  $(b_1, \dots, b_k)$  is in  $\mathbb{N}_0^k$  and each  $\geq_i$  is either  $\geq$  or equality  $=$ ; we call such  $P$  *simple sets*. It appears that this generalization of Theorem 2.24 provides a unified explanation of the exact polynomial results in Theorems 1.1, 2.4, 2.11, and 2.17, by mapping downsets of structures in a size-preserving manner onto simple sets in  $\mathbb{N}_0^k$  for some  $k$ .

## 2.4 Growths of profiles of relational structures

In this subsection we mostly follow the survey article of Pouzet [99], see also Cameron [46]. This approach of relational structures was pioneered by Fraïssé [64, 65, 66]. A *relational structure*  $R = (X, (R_i \mid i \in I))$  on  $X$  is formed by the underlying set  $X$  and relations  $R_i \subset X^{m_i}$  on  $X$ ; the sets  $X$  and  $I$  may be infinite. The *size* of  $R$  is the cardinality of  $X$  and  $R$  is called *finite* (*infinite*) if  $X$  is finite (*infinite*). The *signature* of  $R$  is the list  $(m_i \mid i \in I)$  of arities  $m_i \in \mathbb{N}_0$  of the relations  $R_i$ . It is *bounded* if the numbers  $m_i$  are bounded and is *finite* if  $I$  is finite.

Consider two relational structures  $R = (X, (R_i \mid i \in I))$  and  $S = (Y, (S_i \mid i \in I))$  with the same signature and an injection  $f : X \rightarrow Y$  satisfying, for every  $i$  in  $I$  and every  $m_i$ -tuple  $(a_1, a_2, \dots, a_{m_i})$  in  $X^{m_i}$ ,

$$(a_1, a_2, \dots, a_{m_i}) \in R_i \iff (f(a_1), f(a_2), \dots, f(a_{m_i})) \in S_i.$$

If such an injection  $f$  exists, we say that  $R$  is *embeddable* in  $S$  and write  $R \prec S$ . If in addition  $f$  is an identity (in particular,  $X \subset Y$ ),  $R$  is a *substructure* of  $S$  and we write  $R \prec^* S$ . If the injection  $f$  is onto  $Y$ , we say that  $R$  and  $S$  are *isomorphic*.

The *age* of a (typically infinite) relational structure  $R$  on  $X$  is the set  $P$  of all finite substructures of  $R$ . Note that the age forms a downset in the poset  $(U, \prec^*)$  of all finite relational structures with the signature of  $R$  whose underlying sets are subsets of  $X$ . The *kernel* of  $R$  is the set of elements  $x$  in  $X$  such that the deletion of  $x$  changes the age. The *profile* of  $R$  is the unlabeled counting function  $g_R(n)$  that counts nonisomorphic structures with size  $n$  in the age of  $R$ . We get the same function if we replace the age of  $R$  by the set  $P$  of all finite substructures embeddable in  $R$  whose underlying sets are  $[n]$ :

$$g_R(n) = \#(\{S = ([n], (S_i \mid i \in I)) \mid S \prec R\} / \sim)$$

where  $\sim$  is the isomorphism relation.

The next general result on growth of profiles was obtained by Pouzet [97], see also [99].

**Theorem 2.25** (Pouzet). *If  $g_R(n)$  is the profile of an infinite relational structure  $R$  with bounded signature or with finite kernel, then exactly one of the three cases occurs.*

1. *The function  $g_R(n)$  is constant for large  $n$ .*
2. *There are constants  $k$  in  $\mathbb{N}$  and  $0 < c < d$  such that  $cn^k < g_R(n) < dn^k$  for every  $n$ .*
3. *One has  $g_R(n) > n^k$  for large  $n$  for every constant  $k$  in  $\mathbb{N}$ .*

Case 1 follows from the interesting fact that every infinite  $R$  has a nondecreasing profile (Pouzet [98], see [99, Theorem 4] for further discussion) and cases 2 and 3 were proved in [97] (see [99, Theorems 7 and 42]). It is easy to see ([99, Theorem 10]) that for unbounded signature one can get arbitrarily slowly growing unbounded profiles. Also, it turns out ([99, Fact 2]) that for bounded signature and finite-valued profile, one may assume without loss of generality that the signature is finite. An infinite graph  $G = (\mathbb{N}, E)$  whose components are cliques and that for every  $n$  has infinitely many components (cliques) of size  $n$  shows that for the signature (2) the numbers of integer partitions  $p_n$  appear as a profile (cf. Theorem 2.16)—for relational structures in general (and unlabeled count) there is no polynomial-exponential jump (but cf. Theorem 2.21).

The survey [99] contains, besides further results and problems on profiles of relational structures, the following attractive conjecture which was partially resolved by Pouzet and Thiéry [100].

**Problem 2.26.** *In the cases 1 and 2 of Theorem 2.25 function  $g_R(n)$  is a quasipolynomial for large  $n$ .*

As remarked in [99], since  $g_P(n)$  is nondecreasing, if the conjecture holds then the leading coefficient in the quasipolynomial must be constant and, in cases 1 and 2,  $g_R(n) = an^k + O(n^{k-1})$  for some constants  $a > 0$  and  $k$  in  $\mathbb{N}_0$  (cf. Theorem 2.16 and the following comment).

Relational structures are quite general in allowing arbitrarily many relations with arbitrary arities and therefore they can accommodate many previously discussed combinatorial structures and many more. On the other hand, ages of relational structures are less general than downsets of structures, every age is a downset in the substructure ordering but not vice versa—many downsets of finite structures do not come from a single infinite structure (a theorem due to Fraïssé [99, Lemma 7], [46] characterizes downsets that are ages). An interesting research direction may be to join the general sides of both approaches.

### 3 Five topics in general enumeration

In this section we review five topics in general enumeration. As we shall see, there are connections to the results on growth of downsets presented in the previous section.

#### 3.1 Counting lattice points in polytopes

A *polytope*  $P$  in  $\mathbb{R}^k$  is a convex hull of a finite set of points. If these points have rational, respectively integral, coordinates, we speak of *rational*, respectively *lattice*, polytope. For a polytope  $P$  and  $n$  in  $\mathbb{N}$  we consider the dilation  $nP = \{nx \mid x \in P\}$  of  $P$  and the number of lattice points in it,

$$f_P(n) = \#(nP \cap \mathbb{Z}^k).$$

The following useful result was derived by Ehrhart [56] and Macdonald [85, 86].

**Theorem 3.1** (Ehrhart, Macdonald). *If  $P$  is a lattice polytope, respectively rational polytope, and  $f_P(n)$  counts lattice points in the dilation  $nP$ , then there is a rational polynomial, respectively rational quasipolynomial,  $p(x)$  such that  $f_P(n) = p(n)$  for every  $n$ .*

For further refinements and ramifications of this result and its applications see Beck and Robins [25] (also Stanley [114]). Barvinok [23] and Barvinok and Woods [24] developed a beautiful and powerful theory producing polynomial-time algorithms for counting lattice points in rational polytopes. In way of specializations one obtains from it many Wilfian formulas. We will not say more on it because in its generality it is out of scope of this overview (as we said, this not a survey on  $\#P$ ).



### 3.2 Context-free languages

A *language*  $P$  is a subset of  $A^*$ , the infinite set of finite words over a finite alphabet  $A$ . The natural size function  $|\cdot|$  measures length of words and

$$f_P(n) = \#\{u \in P \mid |u| = n\}$$

is the number of words in  $P$  with length  $n$ . In this subsection the alphabet  $A$  is always finite, thus  $f_P(n) \leq |A|^n$ .

We review the definition of context-free languages, for further information on (formal) languages see Salomaa [103]. A *context-free grammar* is a quadruple  $G = (A, B, c, D)$  where  $A, B$  are finite disjoint sets,  $c \in B$  (starting variable) and  $D$  (production rules) is a finite set of pairs  $(d, u)$  where  $d \in B$  and  $u \in (A \cup B)^*$ . A *rightmost derivation* of a word  $v \in (A \cup B)^*$  in  $G$  is a sequence of words  $v_1 = c, v_2, \dots, v_r = v$  in  $(A \cup B)^*$  such that  $v_i$  is obtained from  $v_{i-1}$  by replacing the rightmost occurrence of a letter  $d$  from  $B$  in  $v_{i-1}$  by the word  $u$ , according to some production rule  $(d, u) \in D$ . (Note that no  $v_i$  with  $i < r$  is in  $A^*$ .) We let  $L(G)$  denote the set of words in  $A^*$  that have rightmost derivation in  $G$ . If in addition every  $v$  in  $L(G)$  has a unique rightmost derivation in  $G$ , then  $G$  is an *unambiguous context-free grammar*. A language  $P \subset A^*$  is *context-free* if  $P = L(G)$  for a context-free grammar  $G = (A, B, c, D)$ .  $P$  is, in addition, *unambiguous* if it can be generated by an unambiguous context-free grammar. If  $P$  is context-free but not unambiguous, we say that  $P$  is *inherently ambiguous*. We associate with a context-free grammar  $G = (A, B, c, D)$  a digraph  $H(G)$  on the vertex set  $B$  by putting an arrow  $d_1 \rightarrow d_2$ ,  $d_i \in B$ , if and only if there is a production rule  $(d_1, u) \in D$  such that  $d_2$  appears in  $u$ . We call a context-free language *ergodic* if it can be generated by a context-free grammar  $G$  such that the digraph  $H(G)$  is strongly connected.

Chomsky and Schützenberger [51] obtained the following important result.

**Theorem 3.2** (Chomsky and Schützenberger). *If  $P$  is an unambiguous context-free language and  $f_P(n)$  counts words of length  $n$  in  $P$ , then the generating function*

$$F(x) = \sum_{n \geq 0} f_P(n)x^n$$

*of  $P$  is algebraic over  $\mathbb{Q}(x)$ .*

The algebraicity of a power series  $F(x) = \sum_{n \geq 0} a_n x^n$  with  $a_n$  in  $\mathbb{N}_0$  has two important practical corollaries for the counting sequence  $(a_n)_{n \geq 1}$ . First, as we already mentioned, it is holonomic. Second, it has a nice asymptotics. More precisely,  $F(x)$  determines a function analytic in a neighborhood of 0 and if  $F(x)$  is not a polynomial, it has a finite radius of convergence  $\rho$ ,  $0 < \rho \leq 1$ , and finitely

many (dominating) singularities on the circle of convergence  $|x| = \rho$ . In the case of single dominating singularity we have

$$a_n \sim cn^\alpha r^n$$

where  $c > 0$  is in  $\mathbb{R}$ ,  $r = 1/\rho \geq 1$  is an algebraic number, and the exponent  $\alpha$  is in  $\mathbb{Q} \setminus \{-1, -2, -3, \dots\}$  (if  $F(x)$  is rational, then  $\alpha$  is in  $\mathbb{N}_0$ ). For example, for Catalan numbers  $C_n = \frac{1}{n+1} \binom{2n}{n}$  and their generating function  $C(x) = \sum_{n \geq 0} C_n x^n$  we have

$$xC^2 - C + 1 = 0 \quad \text{and} \quad C_n \sim \pi^{-1/2} n^{-3/2} 4^n.$$

For more general results on asymptotics of coefficients of algebraic power series see Flajolet and Sedgewick [63, Chapter VII].

Flajolet [62] used Theorem 3.2 to prove the inherent ambiguity of certain context-free languages. For further information on rational and algebraic power series in enumeration and their relation to formal languages see Barucci et al. [22], Bousquet-Mélou [39, 40], Flajolet and Sedgewick [63] and Salomaa and Soittola [104].

How fast do context-free languages grow? Trofimov [116] proved for them a polynomial to exponential jump.

**Theorem 3.3** (Trofimov). *If  $P \subset A^*$  is a context-free language over the alphabet  $A$ , then either  $f_P(n) \leq |A|n^k$  for every  $n$  or  $f_P(n) > c^n$  for large  $n$ , where  $k > 0$  and  $c > 1$  are constants.*

Trofimov proved that in the former case in fact  $P \subset w_1^* w_2^* \dots w_k^*$  for some  $k$  words  $w_i$  in  $A^*$ . Later this theorem was independently rediscovered by Incitti [74] and Bridson and Gilman [41]. D'Alessandro, Intrigila and Varricchio [7] show that in the former case the function  $f_P(n)$  is in fact a quasipolynomial  $p(n)$  for large  $n$  (and that  $p(n)$  and the bound on  $n$  can be effectively determined from  $P$ ).

Recall that for a language  $P \subset A^*$  the growth constant is defined as  $c(P) = \limsup f_P(n)^{1/n}$ .  $P$  is *growth-sensitive* if  $c(P) > 1$  and  $c(P \cap Q) < c(P)$  for every downset  $Q$  in  $(A^*, \prec)$ , where  $\prec$  is the subword ordering, such that  $P \cap Q \neq P$ . In other words, forbidding any word  $u$  such that  $u \prec v \in P$  for some  $v$  as a subword results in a significant decrease in growth. Yet in other words, in growth-sensitive languages an analogue of Marcus–Tardos theorem (Theorem 2.2) holds. In a series of papers Ceccherini-Silberstein, Machì and Scarabotti [48], Ceccherini-Silberstein and Woess [49, 50], and Ceccherini-Silberstein [47] the following theorem on growth-sensitivity was proved.

**Theorem 3.4** (Ceccherini-Silberstein, Machì, Scarabotti, Woess). *Every unambiguous context-free language  $P$  that is ergodic and has  $c(P) > 1$  is growth-sensitive.*

See [49] for extensions of the theorem to the ambiguous case and [48] for the more elementary case of regular languages.

### 3.3 Exact counting of regular and other graphs

We consider finite simple graphs and, for a given set  $P \subset \mathbb{N}_0$ , the counting function ( $\deg(v) = \deg_G(v)$  is the degree of a vertex  $v$  in  $G$ , the number of incident edges)

$$f_P(n) = \#\{G = ([n], E) \mid \deg_G(v) \in P \text{ for every } v \text{ in } [n]\}.$$

For example, for  $P = \{k\}$  we count labeled  $k$ -regular graphs on  $[n]$ . The next general theorem was proved by Gessel [67, Corollary 11], by means of symmetric functions in infinitely many variables.

**Theorem 3.5** (Gessel). *If  $P$  is a finite subset of  $\mathbb{N}_0$  and  $f_P(n)$  counts labeled graphs on  $[n]$  with all degrees in  $P$ , then the sequence  $(f_P(n))_{n \geq 1}$  is holonomic.*

This theorem was conjectured and partially proved for the  $k$ -regular case,  $k \leq 4$ , by Goulden and Jackson [68]. As remarked in [67], the theorem holds also for graphs with multiple edges and/or loops. Domocoş [54] extended it to 3-regular and 3-partite hypergraphs (and remarked that Gessel's method works also for general  $k$ -regular and  $k$ -partite hypergraphs). For more information see also Mishna [92, 93].

Consequently, the numbers of labeled graphs with degrees in fixed finite set have Wilfian formula. Now we demonstrate this directly by a more generally applicable argument. For  $d$  in  $\mathbb{N}_0^{k+1}$ , we say that a graph  $G$  is a  $d$ -graph if  $|V(G)| = d_0 + d_1 + \dots + d_k$ ,  $\deg(v) \leq k$  for every  $v$  in  $V(G)$  and exactly  $d_i$  vertices in  $V(G)$  have degree  $i$ . Let

$$p(d) = p(d_0, d_1, \dots, d_k) = \#\{G = ([n], E) \mid G \text{ is a } d\text{-graph}\}$$

be the number of labeled  $d$ -graphs with vertices  $1, 2, \dots, n = d_0 + \dots + d_k$ .

**Proposition 3.6.** *For fixed  $k$ , the list of numbers*

$$(p(d) \mid d \in \mathbb{N}_0^{k+1}, d_0 + d_1 + \dots + d_k = m \leq n)$$

*can be generated in time polynomial in  $n$ .*

**Proof.** A natural idea is to construct graphs  $G$  with  $d_i$  vertices of degree  $i$  by adding vertices  $1, 2, \dots, n$  one by one, keeping track of the numbers  $d_i$ . In the first phase of the algorithm we construct an auxiliary  $(n+1)$ -partite graph

$$H = (V_0 \cup V_1 \cup \dots \cup V_n, E)$$

where we start with  $V_m$  consisting of all  $(k+1)$ -tuples  $d = (d_0, d_1, \dots, d_k)$  in  $\mathbb{N}_0^{k+1}$  satisfying  $d_0 + \dots + d_k = m$  and the edges will go only between  $V_m$  and  $V_{m+1}$ . An edge joins  $d \in V_m$  with  $e \in V_{m+1}$  if and only if there exist numbers  $\Delta_0, \Delta_1, \dots, \Delta_{k-1}$  in  $\mathbb{N}_0$  such that:  $0 \leq \Delta_i \leq d_i$  for  $0 \leq i \leq k-1$ ,  $r := \Delta_0 + \dots + \Delta_{k-1} \leq k$ , and

$$e_i = d_i + \Delta_{i-1} - \Delta_i \text{ for } i \in \{0, 1, \dots, k\} \setminus \{r\} \text{ but } e_r = d_r + \Delta_{r-1} - \Delta_r + 1$$

where we set  $\Delta_{-1} = \Delta_k = 0$ . We omit from  $H$  (or better, do not construct at all) the vertices  $d$  in  $V_m$  not reachable from  $V_0 = \{(0, 0, \dots, 0)\}$  by a path  $v_0, v_1, \dots, v_m = d$  with  $v_i$  in  $V_i$ . For example, the  $k$  vertices in  $V_1$  with  $d_0 = 0$  are omitted and only  $(1, 0, \dots, 0)$  remains. Also, we label the edge  $\{e, d\}$  with the  $k$ -tuple  $\Delta = (\Delta_0, \dots, \Delta_{k-1})$ . (It follows that  $\Delta$  and  $r$  are uniquely determined by  $d, e$ .) The graph  $H$  together with its labels can be constructed in time polynomial in  $n$ . It records the changes of the numbers  $d_i$  of vertices with degree  $i$  caused by adding to  $G = ([m], E)$  new vertex  $m+1$ ;  $\Delta_i$  are the numbers of neighbors of  $m+1$  with degree  $i$  in  $G$  and  $r$  is the degree of  $m+1$ .

In the second phase we evaluate a function  $p : V_0 \cup \dots \cup V_n \rightarrow \mathbb{N}$  defined on the vertices of  $H$  by this inductive rule:  $p(0, 0, \dots, 0) = 1$  on  $V_0$  and, for  $e$  in  $V_m$  with  $m > 0$ ,

$$p(e) = \sum_d p(d) \prod_{i=0}^{k-1} \binom{d_i}{\Delta_i}$$

where we sum over all  $d$  in  $V_{m-1}$  such that  $\{d, e\} \in E(H)$  and  $\Delta$  is the label of the edge  $\{d, e\}$ . It is easy to see that all values of  $p$  can be obtained in time polynomial in  $n$  and that  $p(d)$  for  $d$  in  $V_m$  is the number of labeled  $d$ -graphs on  $[m]$ .  $\square$

Now we can in time polynomial in  $n$  easily calculate

$$f_P(n) = \sum_d p(d)$$

as a sum over all  $d = (d_0, \dots, d_k)$  in  $V_n$ ,  $k = \max P$ , satisfying  $d_i = 0$  when  $i \notin P$ . Of course, this algorithm is much less effective than the holonomic recurrence ensured by (and effectively obtainable by the proof of) Theorem 3.5. But by this approach we can get Wilfian formula also for some infinite sets of degrees  $P$ , for example when  $P$  is an arithmetical progression. (We leave to the reader as a nice exercise to count labeled graphs with even degrees.) On the other hand, it seems to fail for many classes of graphs, for example, for triangle-free graphs.

**Problem 3.7.** *Is there a Wilfian formula for the number of labeled triangle-free graphs on  $[n]$ ? Can this number be calculated in time polynomial in  $n$ ?*

The problem of enumeration of labeled triangle-free graphs was mentioned by Read in [101, Chapter 2.10]. A quarter century ago, Wilf [119] posed the following similar problem.

**Problem 3.8.** *Can one calculate in time polynomial in  $n$  the number of unlabeled graphs on  $[n]$ ?*

### 3.4 The ubiquitous asymptotics $cn^{-3/2}r^n$

In many enumerative problems about recursively defined structures, for example when counting rooted trees of various kinds, one ends up with asymptotic form of the type

$$f_P(n) \sim cn^{-3/2}r^n$$

where  $c > 0$  and  $r > 1$  are constants. Bell, Burris and Yeats [26] developed a remarkable general theory explaining ubiquity of this asymptotic relation. Their results give practical and general tool for proving this asymptotics by checking certain conditions for the operator on power series  $\Theta$  that appears in the equation  $F = \Theta(F)$  expressing the counting problem in terms of the generating function  $F = \sum_{n \geq 0} f_P(n)x^n$ . These conditions are often easy to check; the reader is referred to [26] for examples. We state their main result [26, Theorem 75] below as Theorem 3.9. For the complete statement of the theorem it is necessary to introduce several notions. Eventually we define two classes  $\mathcal{O}_E$  and  $\mathcal{O}_I$  of operators on power series guaranteeing the asymptotics  $cn^{-3/2}r^n$ .

By  $\mathbb{R}[[x, y]]_{\geq 0}$  we denote the set of bivariate power series with nonnegative real coefficients and zero constant term, and similarly for  $\mathbb{R}[[x]]_{\geq 0}$  and  $\mathbb{Z}[[x]]_{\geq 0}$ . An *operator* is a mapping  $\Theta$  from  $\mathbb{R}[[x]]_{\geq 0}$  to itself. It is *integral* if it preserves  $\mathbb{Z}[[x]]_{\geq 0}$ . We say that  $\Theta$  is a *retro operator* if for every  $n$  the coefficient  $[x^n]\Theta(F)$  depends only on the coefficients  $[x^m]F$  with  $m < n$ , in particular,  $[x]\Theta(F)$  is a constant independent of  $F$ . *Elementary operator*  $\Theta$  is given by a power series  $E(x, y)$  in  $\mathbb{R}[[x, y]]_{\geq 0}$  by  $\Theta(F) = E(x, F)$ . We also say that  $E$  *represents*  $\Theta$ . An elementary operator  $\Theta = E$  is *nonlinear* if  $E(x, y)$  is nonlinear in  $y$ , is *bounded* if  $[x^n]E(x, x) < c^n$  for every  $n$  and a constant  $c > 1$ , and is *open* if for every  $a, b > 0$  the convergence  $E(a, b) < +\infty$  implies the convergence  $E(a + \varepsilon, b + \varepsilon) < +\infty$  for some  $\varepsilon > 0$ .  $\mathcal{O}_E$  is the set of bounded and open elementary operators. By  $\mathcal{O}_E^*$  we denote the subset of integral operators in  $\mathcal{O}_E$ .

The second set of operators  $\mathcal{O}_I$  is defined as a closure, under the operation of scalar multiplication by positive reals and the binary operations of addition (+), multiplication ( $\cdot$ ), and composition ( $\circ$ ) of operators, of the set of *base operators*

$$\mathcal{O}_E^* \cup \{\Theta_{H,M} \mid H \in \{S, D, C, I\}, M \subset \mathbb{N}\}$$

where for  $H \in \{D, C\}$  we allow only finite sets  $M$  or sets satisfying  $\sum_{m \in M} 1/m = +\infty$ . The operators  $\Theta_{H,M}$  are defined as follows. For a permutational group  $H$  acting on  $[m]$ , Pólya's cycle index polynomial is

$$Z(H, x_1, \dots, x_m) = \frac{1}{|H|} \sum_{\sigma \in H} \prod_{j=1}^m x_j^{\sigma_j}$$

where  $\sigma_j$  is the number of  $j$ -cycles in the decomposition of  $\sigma$  into disjoint cycles. In the role of  $H$  we take four families of permutational groups:  $S_m$  (symmetric group of order  $m!$ ),  $D_m$  (dihedral group of order  $2m$ ),  $C_m$  (cyclic group of order  $m$ ) and  $I_m$  (identity group of order 1). For  $m$  in  $\mathbb{N}$  we define four operators  $\Theta_{S,m}$ ,  $\Theta_{D,m}$ ,  $\Theta_{C,m}$ , and  $\Theta_{I,m}$ , by

$$\Theta_{S,m}(F) = Z(S_m, F(x), F(x^2), \dots, F(x^m)),$$

and by analogous expressions for the other three operators. For  $M \subset \mathbb{N}$  we set

$$\Theta_{S,M} = \sum_{m \in M} \Theta_{S,m}$$

and similarly for the cases  $D, C, I$ . This completes the definition of the base operators  $\Theta_{S,M}$ ,  $\Theta_{D,M}$ ,  $\Theta_{C,M}$ , and  $\Theta_{I,M}$ . Note that each of them, as well as each operator in  $\mathcal{O}_E^*$ , is integral. It follows that every operator in  $\mathcal{O}_I$  is integral.

The base operators  $\Theta_{S,M}$ ,  $\Theta_{D,M}$ ,  $\Theta_{C,M}$ , and  $\Theta_{I,M}$  correspond, respectively, to the combinatorial construction of multiset, cycle, directed cycle and sequence, with  $M$  listing the allowed numbers of components. For example, the sequence operator  $\Theta_{I,M}$  is simply the elementary integral operator  $E(x, y) = \sum_{m \in M} y^m$ . See Flajolet and Sedgewick [63] for these construction and their relations to generating functions, and Bergeron, Labelle and Leroux [27] for a more abstract and general approach.

Following the inductive definition of  $\mathcal{O}_I$ , one associates with a given  $\Theta$  in  $\mathcal{O}_I$  and  $F$  in  $\mathbb{R}[[x]]_{\geq 0}$  a canonical power series  $E(x, y) = E^{\Theta, F}(x, y)$  in  $\mathbb{R}[[x, y]]_{\geq 0}$  representing  $\Theta$  at  $F$ , which means that  $\Theta(F) = E^{\Theta, F}(x, F)$ . We say that  $\Theta$  in  $\mathcal{O}_I$  is *nonlinear* if  $E^{\Theta, x}(x, y)$  is nonlinear in  $y$ .

We are ready to state the theorem. The lower indices in  $E_x$  and  $E_{yy}$  indicate partial derivatives.

**Theorem 3.9** (Bell, Burris and Yeats). *Suppose that  $\Theta$  is a nonlinear retro operator in  $\mathcal{O}_E$ , respectively in  $\mathcal{O}_I$ , and that the power series  $A$  in  $\mathbb{R}[[x]]_{\geq 0}$ , respectively in  $\mathbb{Z}[[x]]_{\geq 0}$ , diverges in its radius of convergence. Then the equation*

$$F = A + \Theta(F)$$

has a unique solution  $F$  in  $\mathbb{R}[[x]]_{\geq 0}$ , respectively in  $\mathbb{Z}[[x]]_{\geq 0}$ , and the coefficients of  $F$  have the following asymptotic form. There are constants  $d \in \mathbb{N}_0$ ,  $q \in \mathbb{N}$ ,  $c > 0$ , and  $r > 1$  such that  $[x^n]F = 0$  for  $n \not\equiv d \pmod q$  but

$$[x^n]F \sim cn^{-3/2}r^n$$

as  $n \rightarrow \infty$  for  $n \equiv d \pmod q$ . Moreover,  $r = 1/\rho$  where  $\rho$  is the radius of convergence of  $F$  and

$$c = q \cdot \sqrt{\frac{\rho \cdot E_x(\rho, F(\rho))}{2\pi \cdot E_{yy}(\rho, F(\rho))}}$$

where  $E = E(x, y)$  in  $\mathbb{R}[[x, y]]_{\geq 0}$  represents the operator  $L : G \mapsto A + \Theta(G)$ , respectively  $E$  represents  $L$  at  $F$ .

An important combinatorial construction left out from the definition of  $\mathcal{O}_I$  is the set construction, with corresponding operators  $\Theta_{\pm S, M}$  and  $\Theta_{\pm S, m}$  where

$$\Theta_{\pm S, m}(F) = Z(S_m, F(x), -F(x^2), F(x^3), \dots, (-1)^{m+1}F(x^m)).$$

Negative signs are troublesome because the nonnegativity of coefficients of power series is crucial at several steps of proof of Theorem 3.9. Besides other interesting problems, the authors of [26] pose the following one ([26, Q2 in Section 8], [26, Section 6.2]).

**Problem 3.10.** *Can the set operator  $\Theta_{\pm S, M}$  be adjoined to the base operators in the definition of  $\mathcal{O}_I$  so that the universal asymptotics  $cn^{-3/2}r^n$  still follows?*

### 3.5 Ultimate modular periodicity

One general aspect of counting functions  $f_P(n)$  not touched so far is their modular behavior. For given modulus  $m$  in  $\mathbb{N}$ , what can be said about the sequence of residues  $(f_P(n) \pmod m)_{n \geq 1}$ . Before presenting a rather general result in this area, we motivate it by two examples.

First, Bell numbers  $B_n$  counting partitions of  $[n]$ . Recall that

$$\sum_{n \geq 0} B_n x^n = \sum_{k=0}^{\infty} \frac{x^k}{(1-x)(1-2x) \dots (1-kx)}.$$

Reducing modulo  $m$  we get, denoting  $v(x) = (1-x)(1-2x)\dots(1-(m-1)x)$ ,

$$\begin{aligned}
\sum_{n \geq 0} B_n x^n &\equiv_m \sum_{j=0}^{\infty} \frac{x^{mj}}{v(x)^j} \sum_{i=0}^{m-1} \frac{x^i}{(1-x)(1-2x)\dots(1-ix)} \\
&= \frac{1}{1-x^m/v(x)} \sum_{i=0}^{m-1} \frac{x^i}{(1-x)(1-2x)\dots(1-ix)} \\
&= \frac{a(x)}{v(x) - x^m} = \frac{a(x)}{1 + b_1x + \dots + b_mx^m} \\
&= \sum_{n \geq 0} c_n x^n
\end{aligned}$$

where  $a(x) \in \mathbb{Z}[x]$  has degree at most  $m-1$  and  $b_i$  are integers,  $b_m = -1$ . Thus the sequence of integers  $(c_n)_{n \geq 1}$  satisfies for  $n > m$  the linear recurrence  $c_n = -b_1c_{n-1} - \dots - b_mc_{n-m}$  of order  $m$ . By the pigeonhole principle, the sequence  $(c_n \bmod m)_{n \geq 1}$  is periodic for large  $n$ . Since  $B_n \equiv c_n \bmod m$ , the sequence  $(B_n \bmod m)_{n \geq 1}$  is periodic for large  $n$  as well. (For modular periods of Bell numbers see Lunnon, Pleasants and Stephens [84]).

Second, Catalan numbers  $C_n = \frac{1}{n+1} \binom{2n}{n}$  counting, for example, noncrossing partitions of  $[n]$ . The shifted version  $D_n = C_{n-1} = \frac{1}{n} \binom{2n-2}{n-1}$  satisfies the recurrence  $D_1 = 1$  and, for  $n > 1$ ,

$$D_n = \sum_{i=1}^{n-1} D_i D_{n-i} = 2 \sum_{i=1}^{\lfloor n/2 \rfloor - 1} D_i D_{n-i} + \sum_{i=\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} D_i D_{n-i}.$$

Thus, modulo 2,  $D_1 \equiv 1$ ,  $D_n \equiv 0$  for odd  $n > 1$  and  $D_n \equiv D_{n/2}^2 \equiv D_{n/2}$  for even  $n$ . It follows that  $D_n \equiv 1$  if and only if  $n = 2^m$  and that the sequence  $(C_n \bmod 2)_{n \geq 1}$  has 1's for  $n = 2^m - 1$  and 0's elsewhere. In particular, it is not periodic for large  $n$ . (For modular behavior of Catalan numbers see Deutsch and Sagan [53] and Eu, Liu and Yeh [58]).

Bell numbers come out as a special case of a general setting. Consider a *relational system* which is a set  $P$  of relational structures  $R$  with the same finite signature and underlying sets  $[n]$  for  $n$  ranging in  $\mathbb{N}$ . We say that  $P$  is *definable in MSOL*, monadic second-order logic, if  $P$  coincides with the set of finite models (on sets  $[n]$ ) of a closed formula  $\phi$  in MSOL. (MSOL has, in addition to the language of the first-order logic, variables  $S$  for sets of elements, which can be quantified by  $\forall, \exists$ , and atomic formulas of the type  $x \in S$ ; see Ebbinghaus and Flum [55].) Let  $f_P(n)$  be the number of relational structures in  $P$  on the set  $[n]$ , that is, the number of models of  $\phi$  on  $[n]$  when  $P$  is defined by  $\phi$ . For example, the (first-order)



formula  $\phi$  given by ( $a, b, c$  are variables for elements,  $\sim$  is a binary predicate)

$$\forall a, b, c : (a \sim a) \ \& \ (a \sim b \Rightarrow b \sim a) \ \& \ ((a \sim b \ \& \ b \sim c) \Rightarrow a \sim c)$$

has as its models equivalence relations and  $f_P(n) = f_\phi(n) = B_n$ , the Bell numbers.

Let us call a sequence  $(s_1, s_2, \dots)$  *ultimately periodic* if it is periodic for large  $n$ : there are constants  $p, q$  in  $\mathbb{N}$  such that  $s_{n+p} = s_n$  whenever  $n \geq q$ . Specker and Blatter ([29, 30, 31], see also Specker [109]) proved the following remarkable general theorem.

**Theorem 3.11** (Specker and Blatter). *If a relational system  $P$  definable in MSOL uses only unary and binary relations and  $f_P(n)$  counts its members on the set  $[n]$ , then for every  $m$  in  $\mathbb{N}$  the sequence  $(f_P(n) \bmod m)_{n \geq 1}$  is ultimately periodic.*

The general reason for ultimate modular periodicity in Theorem 3.11 is the same as in our example with  $B_n$ , residues satisfy a linear recurrence with constant coefficients. Fischer [59] constructed counterexamples to the theorem for quaternary relations, see also Specker [110]. Fischer and Makowski [60] extended Theorem 3.11 to CMSOL (monadic second-order logic with modular counting) and to relations with higher arities when vertices have bounded degrees. Note that regular graphs and triangle-free graphs are first-order definable. Thus the counting sequences mentioned in Theorem 3.5 and in Problem 3.7 are ultimately periodic to any modulus. More generally, any hereditary graph property  $P = \text{Av}(F)$  with finite base  $F$  is first-order definable and similarly for other structures. Many hereditary properties with infinite bases (and also many sets of graphs or structures which are not hereditary) are MSOL-definable; this is the case, for example, for forests ( $P = \text{Av}(F)$  where  $F$  is the set of cycles) and for planar graphs (use Kuratowski's theorem). To all of them Theorem 3.11 applies. On the other hand, as the example with Catalan numbers shows, counting of ordered structures is in general out of reach of Theorem 3.11.

A closely related circle of problems is the determination of spectra of relational systems  $P$  and more generally of finite models; the *spectrum* of  $P$  is the set

$$\{n \in \mathbb{N} \mid f_P(n) > 0\}$$

—the set of sizes of members of  $P$ . In several situation it was proved that the spectrum is an ultimately periodic subset of  $\mathbb{N}$ . See Fischer and Makowski [61] (and the references therein), Shelah [106] and Shelah and Doron [107]. We conclude with a problem posed in [59].

**Problem 3.12.** *Does Theorem 3.11 hold for relational systems with ternary relations?*

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